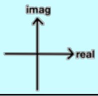
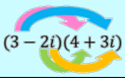
Complex Numbers				
Basics				
Definition Of i $\sqrt{-1} = i$ $i^2 = -1$				
Calculating Powers Of i: Use $i^2 = -1$ and indices rule $(x^a)^b = x^{ab}$ $i^{48} = (i^2)^{24} = (-1)^{24} = 1$ $i^{27} = (i^2)^{13} i = (-1)i = -i$				
Calculating Square Roots $\sqrt{-10}\sqrt{-40} = \sqrt{-1}\sqrt{10}\sqrt{-1}\sqrt{40} = i\sqrt{10}i\sqrt{40} = i^2\sqrt{400} = -20$. Do not make the mistake of saying $\sqrt{-10}\sqrt{-40} = \sqrt{400} = 20$				
Cartesian Form				
Cartesian Form We usually use the letter z to denote a complex number $z =$ real part $\pm$ i (imaginary part) Note: This can also be written as $z =$ real part $\pm$ (imaginary part) i We usually use the letters a and b or x and y $z = a + ib$ or $z = x + iy$ $Re(z)$ means the real part of z $Im(z)$ means the imaginary part of z				
Complex Conjugate We swap the sign of the imaginary part for the complex conjugate $z = a + bi \Rightarrow z^* = a - bi$ $z = a - bi \Rightarrow z^* = a + bi$ We use the notation z^* *or* $\bar{z}$ to denote the complex conjugate of z.				
Representation On An Argand Diagram  				
Adding and Subtracting When we add/subtract we combine the real parts together and the imaginary parts $(a + bi) + (c + di) = (a + c) + (b + d)i$				
Multiplying When we **multiply**, we expand the brackets as normal and then collect like terms. Remember that i^2 can be replaced with -1  $(3-2i)(4+3i) = 12 + 9i - 8i - 6i^2 = 12 - 6(-1) + i = 12 + 6 + i = 18 + i$				
Dividing Multiply numerator and denominator by the complex conjugate. $\frac{a+bi}{c+di} = \frac{a+bi}{c+di} \times \frac{c-di}{c-di} = \frac{ac-adi+bci-bdi^2}{c^2-cdi+cdi-d^2i^2} = \frac{ac-bdi^2-adi+bci}{c^2-d^2i^2}$ the middle terms cancel out				
Finding The Modulus And Argument The modulus of a complex number denoted, $\lvert z \rvert$, is the distance from the origin to that number on an argand diagram. The argument of a complex number, arg z, is the angle between the positive real axis and the line joining the number to the origin on an Argand diagram. Method to calculate modulus and argument: Given $z = x + yi \Rightarrow \begin{cases} \text{modulus} = \lvert z \rvert = \sqrt{x^2 + y^2} \\ \text{argument} = \arg z = tan^{-1}\left(\frac{+b}{+a}\right) \text{ where } -\pi < \theta < \pi \end{cases}$ Note: The red parts are always a plus Next step for argument: Draw $x + yi$ out to know which quadrant you're in, start from positive x axis and find the anti-clockwise angle to find the value of theta				
Properties Of Modulus And Complex Conjugate: • $\lvert z_1 z_2 \rvert = \lvert z_1 \rvert \lvert z_2 \rvert$ • $\left\lvert \frac{z_1}{z_2} \right\rvert = \frac{\lvert z_1 \rvert}{\lvert z_2 \rvert}$ • $(z \pm w)^* = z^* \pm w^*$ • $(zw)^* = z^* w^*$ • $\left(\frac{z}{w}\right)^* = \frac{z^*}{w^*}$ if $w \neq 0$ • $z \times z^* = \lvert z \rvert^2$				
Use of **factor theorem** and **polynomial division**				
Factorising Quadratics: Use quadratic formula and work backwards				

Understanding Roots of Quadratics and Cubics:
• A cubic with real coefficients either has:
  ○ all three roots real
  ○ one root real and the other two form a complex conjugate pair				

• A quartic with real coefficients either has: ◦ all four roots real ◦ two roots real and the other two form a complex conjugate pair ◦ two roots form a conjugate pair and the other two roots also form a conjugate pair				
Factorising Cubics, Quartics and Above: Use the factor theorem to find one of the factors and then use algebraic division or comparing coefficients until we have all the factors/roots. Recall that for 2 roots of the polynomial a and b , then we have factors "$(z-a)$" and "$(z-b)$", and can multiply/expand them to get another factor: $z^2-(a+b)z+(ab)$ In other words when we have 2 roots we can build the equation, $z^2-(\text{sum of roots})z+(\text{product of roots})$ Remember that complex number roots occur in conjugate pairs, so if we know one root, then the conjugate is necessarily another root.				
Solving/Finding Roots Of Quadratics: We can use the quadratic formula. For an equation $az^2+bz+c=0$, we get $z_1=\frac{-b+\sqrt{b^2-4ac}}{2a}$ and $z_2=\frac{-b-\sqrt{b^2-4ac}}{2a}$				
Solving/Finding Roots Of Cubics, Quartics And Above: To solve polynomial equations that may have complex roots, we can use the same approach as above to factorise and then we just go one step further by setting the factors equal to 0 after. Remember that complex number roots occur in conjugate pairs, so if we know one root, then the complex conjugate is necessarily another root.				
Given Some Of The Roots, Find The Equation We must use the fact that complex number roots occur in conjugate pairs, so if we know one root, then the conjugate is necessarily another root and then build the equation with the roots $\boldsymbol{a}$ and $\boldsymbol{b}$ as: $z^2-(a+b)z+(ab)$ Now we can use the comparing coefficients method to find the unknowns				
Equating Real and Imaginary Coefficients In Order To: • Find unknowns in equations • Find square roots • Solve equations				
Proving purely real or purely imaginary				
Modulus Argument Form				
Converting cartesian to modulus argument form: $\boldsymbol{a+bi\rightarrow r(cos\theta+isin\theta)=rcis\,\theta}$ We just need to find $r\ and\ \theta$. To find r and θ we use the formula $a+bi\Rightarrow\begin{cases}r=\sqrt{a^2+b^2}\\\theta=tan^{-1}\left(\left\lvert\frac{b}{a}\right\rvert\right)\end{cases}$ and then draw the angle θ in the quadrant where the complex number $a+bi$ lies Read off θ by starting on the positive x axis (like when you solve for trig using the CAST diagram, but remember: $-\pi\le\theta<\pi$ which means we can only go $180°$ in either a clockwise or anti clockwise direction.				
Representation On An Argand Diagram Length and angle				
Complex Conjugate $z^*=\bar{z}=r(\cos\theta-i\sin\theta)$				
De Moivre's Theorem $(cosx+isinx)^n=\cos nx+isin\,nx$ Useful the following follow-on results: • $z^{-n}=r^{-n}(\cos n\theta-i\ \sin n\theta)$ • $z+\frac{1}{z}=2\cos\theta$ • $z-\frac{1}{z}=2i\sin\theta$ • $z^n+\frac{1}{z^n}=z^n+z^{-n}=2\cos n\theta$. Rearranging $\Rightarrow\cos n\theta=\frac{z^n+z^{-n}}{2}$ • $z^n-\frac{1}{z^n}=z^n-z^{-n}=2i\sin n\theta$. Rearranging $\Rightarrow$ $\sin n\theta=\frac{z^n-z^{-n}}{2i}$				
Multiplying and Dividing <u>Multiplying (multiply the moduli and add the arguments)</u> $[r_1(\cos\theta_1+i\sin\theta_1)][r_2(\cos\theta_2+i\sin\theta_2)]=r_1r_2[\cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2)]=r_1r_2e^{i(\theta_1+\theta_2)}$ <u>Dividing (divide the moduli and subtract the arguments)</u> $\frac{r_1(\cos\theta_1+i\sin\theta_1)}{r_2(\cos\theta_2+i\sin\theta_2)}$ can be divided quickly and is $\frac{r_1}{r_2}[\cos(\theta_1-\theta_2)+i\sin(\theta_1-\theta_2)]$				
Representing $z=r(\cos\theta+i\sin\theta)$ as $z=r(\cos(\theta+2k\pi)+i\sin(\theta+2k\pi))$				
Finding cube roots and above (solutions to $z^n=s$) and representation on an argand diagram (we use De Moivre's Theorem) $z^n=s\Longrightarrow z=r^{\frac{1}{n}}\left(\cos\frac{\theta+2k\pi}{n}+i\sin\frac{\theta+2k\pi}{n}\right)$				
Solutions to $z^n=1$ $z=\cos\frac{2k\pi}{n}+i\sin\frac{2k\pi}{n}$ for $k=1,2,3,4,\ldots n$				
nth roots of unity $1,\omega,\omega^2,\omega^3,\ldots,\omega^{n-1}$ (solutions to $z^n=1$ where n is positive integer)and properties: • $1,\omega,\omega^2,\omega^3,\ldots,\omega^{n-1}$ form the vertices of a regular n-gon with centre at the origin • $1+\omega+\omega^2+\omega^3+\cdots+\omega^{n-1}=0$				
If we know one solution to $z^n=s$ (call it z_1) and the solutions to $z^n=1$ (roots of unity) then the roots of $z^n=s$ are $z_1,z_1\omega,z_1\omega^2,\ldots z_1\omega^{n-1}$				

Euler's Form

Converting Modulus Argument Form To Euler's Form

$$r(cos\theta + isin\theta) = re^{i\theta}$$

Note: If given Cartesian form we must turn it into modulus argument form first

Converting Cartesian Form To Euler's Form: $\boldsymbol{a + bi \rightarrow re^{i\theta}}$

Turn it into modulus argument form and then into Eulers form

Multiplying and Dividing:

Multiplying (multiply the moduli and add the arguments)

$$r_1e^{\theta_1} \times r_2e^{\theta_2} = r_1r_2e^{i(\theta_1+\theta_2)}$$

Dividing (divide the moduli and subtract the arguments)

$$\frac{r_1e^{\theta_1}}{r_2e^{\theta_2}} = \frac{r_1}{r_2}e^{i(\theta_1-\theta_2)}$$

Inequality Properties

- $|Re(z)| \leq |z|$ and $|Im(z)| \leq |z|$
- $|z + w| \leq |z| + |w|$
- $|z + w| \geq |z| - |w|$

Loci

$|z| = k \Rightarrow$ circle centre origin and radius k

$|z| < k \Rightarrow$ Inside of circle centre origin and radius k

$|z| \geq k \Rightarrow$outside of circle including circumference centre origin and radius k

$|z - a| = k \Rightarrow$ circle centre a and radius k

$|z - a| < k \Rightarrow$ Inside of circle centre a and radius k

$|z - a| \geq k \Rightarrow$ outside of circle including circumference centre a and radius k

$|z - a| = |z - b| \Rightarrow$let $z = x + iy$ and use $|a + ib| = \sqrt{a^2 + b^2}$ and see which equation you get

$\arg(z - a) = \theta$ is a line from $x = a$ on the x axis with angle θ from the positive x axis

Trig Powers and Linear Functions

Writing Trig Powers In Terms of Linear Functions Of Trig:

To write powers of *cos* in terms of cos and sin

Use $\left(z + \frac{1}{z}\right)^n = (2cosx)^n$ to write $cos^n x$ in terms of $\cos nx$ and/or $\sin nx$

Do binomial on LHS and then group using $z^n \pm \frac{1}{z^n}$ results using $z^n + \frac{1}{z^n} = 2\cos n\theta$

Use indices rule $(x^n)^m$ to simplify RHS

Rearrange for power term

To write powers of *sin* in terms of cos and sin

$\left(z - \frac{1}{z}\right)^n = (2i\sin x)^n$ to write $sin^n x$ in terms of $\cos nx$ and/or $\sin nx$

Do binomial on LHS and then group using $z^n \pm \frac{1}{z^n}$ results using $z^n - \frac{1}{z^n} = 2i\sin n\theta$

Use indices rules $(x^n)^m$ on RHS

Rearrange for power term

Writing Linear Functions Of Trig In Terms Of Trig Powers:

use $(cosx + isinx)^n = \cos nx + isin\, nx$ to write $\cos nx$ or $\sin nx$ in terms of $sin^m x$ and/or $cos^m x$

Note: The equality is true because of De' Moivres theorem

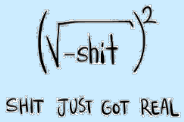
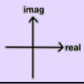
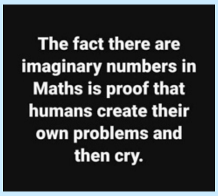
Use binomial expansion on LHS

Equate LHS with the real part of RHS want $cos\ nx$

Equate LHS the imaginary part of RHS want $sin\ nx$

Sum of Series

Use results about sum of geometric series with complex numbers (sum and sum to infinity)

Type	Explanations	Examples
Definition 	$\sqrt{-1}=i$ $i^2=-1$ We are often asked to do calculate powers of i. Relate to i^2 using indices rules to deal with.	**Example 1:** $i^{48}=(i^2)^{24}=(-1)^{24}=1$ **Example 2:** $i^{27}=(i^2)^{13}i=(-1)i=-i$ **Example 3:** $\sqrt{-10}\sqrt{-40}=\sqrt{-1}\sqrt{10}\sqrt{-1}\sqrt{40}=i\sqrt{10}i\sqrt{40}=i^2\sqrt{400}=-20$. Do not make the mistake of saying $\sqrt{-10}\sqrt{-40}=\sqrt{400}=20$
Jargon 	Form: Real + Imaginary so we have $z=x+iy$ with $x,y\in\mathbb{R}$ Note: we can also write $z=x+yi$ • $x=Re(z)$ means the real part of z • $y=Im(z)$ means the imaginary part of z • Modulus $a+bi=\lvert a+bi\rvert=\sqrt{a^2+b^2}$ • Complex conjugate z^* or $\bar{z}=x-iy$ is the complex conjugate of z. Remember if you know one root, then the conjugate is necessarily another root. **Properties:** ○ $(z\pm w)^*=z^*\pm w^*$ ○ $(zw)^*=z^*w^*$ ○ $\left(\frac{z}{w}\right)^*=\frac{z^*}{w^*}$ if $w\neq 0$ ○ $z.z^*=\lvert z\rvert^2$ • Argand diagram 	**Example 1:** Find the complex conjugate, modulus and state the real and imag parts of $2-3i$ The complex conjugate is $2+3i$. 2 is the real part, -3 is the imaginary part Modulus $=\sqrt{2^2+(-3)^2}=\sqrt{4+9}=\sqrt{13}$ **Example 2:** $z=2+3i, w=5-8i$ Find $(z+w)^*$ $(2-3i)+(5+8i)=7+5i$
Adding/Subtracting	• Adding: $(a+ib)+(c+id)=(a+c)+i(b+d)$ • Subtracting: $(a+ib)-(c+id)=(a-c)+i(b-d)$	**Example 1:** $(3-2i)+(4+3i)=(3+4)+(-2+3)i=7+i$ **Example 2:** $(3-2i)-(4+3i)=(3-4)+(-2-3)i=-1-5i$
Multiplying/Dividing	• Multiplying: $(a+ib)(c+id)=ac+adi+bci+i^2bd=(ac-bd)+i(ad+bc)$ • Dividing: $\frac{a+ib}{c+id}$ Multiply by complex conjugate $c-id$ $\frac{a+ib}{c+id}\times\frac{c-id}{c-id}=\frac{ac-adi+bci-i^2bd}{c^2-i^2d^2}=\frac{ac-adi+bci+bd}{c^2+d^2}$ and then simplify further	**Example 1:** $(3-2i)(4+3i)=12+9i-8i-6i^2=12+9i-8i-6(-1)=18+i$ **Example 2:** $\frac{3-2i}{4+3i}=\frac{3-2i}{4+3i}\times\frac{4-3i}{4-3i}=\frac{12-9i-8i-6}{16+9}=\frac{6-17i}{25}=\frac{6}{25}-\frac{17}{25}i$
Solving with Complex Numbers 	We commonly equate real and imaginary parts in order to solve equations with complex numbers in them.	**Example 1:** Find the values of x and y if $(1-i)z=1-3i$ Let $z=x+iy$. $(1-i)(x+iy)=1-3i$ $LHS=x+iy-ix+y=(x+y)+i(-x+y)$ Equating real and imaginary parts gives $x+y=1$ and $-x+y=-3$ $\Longrightarrow x=2, y=-1$ **Example 2:** Given that $\frac{z}{z-8}=-1-2i$, find z in the form $a+ib$ $z=(-1-2i)(z-8)$ $a+ib=(-1-2i)(a+ib-8)$ $a+ib=-a-ib+8-2ai+2b+16i$ $a+ib=(-a+2b+8)+i(-b-2a+16)$ Equating real and imaginary gives $-a+2b+8=-1, -b-2a+16=-2\Longrightarrow a=6, b=2$ $z=6+2i$ **Example 3:** Find the square roots of $8-6i$ Write as $z^2=8-6i\Leftrightarrow z=\sqrt{8-6i}$ $(x+iy)^2=8-6i$ $LHS=x^2+2xyi-y^2=(x^2-y^2)+i(2xy)$ Compare coefficients: $x^2-y^2=8, 2xy=-6$ Solving simultaneously gives $x=\pm3, y=\mp1$, so we get $z=3-i, -3+i$
Factorising & Solving Polynomials 	**Factorising:** Find a factor and then divide by it like usual. **Solving:** ➢ Quadratics: use quadratic formula as usual ➢ Cubics and above: Normally given a root. Remember if you know one root, then the conjugate is necessarily another root. It is quicker to use the equation $x^2-(sum\ roots)x+product\ roots$ instead of writing $(x-root1)(x-root2)$ to build an equation based on the 2 conjugate pair roots that we know. We can then divide by this equation to find further roots. These can be hard for students. For more practice, try the following harder examples after seeing the easier examples on the right. • Factorise the polynomial $P(x)=x^4-5x^3+2x^2+22x-20$ completely with integer coefficients given $3-i$ is a root of $P(x)$ • Given that $(z-1-2i)$ is a factor of $2z^3-3z^2+8z+5$ solve the equation $2z^3-3z^2+8z+5=0$ over the complex number field • Let $P(z)=2z^3+az^2+bz+c$, where $a,b,and\ c\in\mathbb{R}$. Two of the roots of $P(z)$ are -2 and $(-3+2i)$. Find the values of a,b and c (ans a=16, b=50, c=52)	**Example 1:** Solve $z^2+4z+8=0$ $z=\frac{-4\pm\sqrt{4^2-4(1)(8)}}{2(1)}=\frac{-4\pm\sqrt{-16}}{2}=\frac{-4\pm4i}{2}=-2\pm2i$ **Example 2:** Completely factorise $f(x)=x^4-2x^3+2x^2-2x+1$ $f(1)=0$ so dividing by $(x-1)$ gives $g(x)=x^3-x^2+1x-1$ $g(1)=0$ so divide again by $x-1$ which gives x^2+1 $x^4-2x^3+2x^2-2x+1=(x-1)(x-1)(x^2+1)=(x-1)^2(x-i)(x+i)$ **Example 3:** Find all complex numbers z, such that $z^4-z^3+6z^2-z+15=0$ and $z=1+2i$ is a solution to the equation $1-2i$ must be another root since roots occur in conjugate pairs $z^2-(sum\ roots)z+product\ roots=z^2-2z+5$ $(z^4-z^3+6z^2-z+15)\div(z^2-2z+5)=z^2+z+3$ Solve z^2+z+3 $z=\frac{-1\pm\sqrt{1^2-4(1)(3)}}{2(1)}=\frac{-1\pm\sqrt{11}i}{2}=-\frac{1}{2}\pm\frac{\sqrt{11}}{2}i$
Loci	• $\lvert z\rvert=k$ ⇒ circle center origin and radius k • $\lvert z\rvert<k$ ⇒ inside of circle, centered at origin and radius k • $\lvert z\rvert\geq k$ ⇒outside of circle including circumference, centered at origin and radius k • $\lvert z-z_0\rvert=k$ ⇒ is a circle of radius a centered at z_0 • $\lvert z-z_0\rvert<k$ ⇒ inside of circle centered z_0 and radius k • $\lvert z-z_0\rvert\geq k$ ⇒ outside of circle including circumference, centered at z_0 and radius k • $\lvert z-z_0\rvert=\lvert z-z_1\rvert$ To deal with this we let $z=x+iy$ and the take the modulus of each side and see which equation you get. Might be a straight line or circle etc. • $arg\ (z-z_0)=\theta$ is a line from $x=z_0$ on the x axis with angle θ from the positive x axis	Describe clearly the locus in the complex plane defined by the equation $\lvert z+2i\rvert=\lvert 2iz-1\rvert$ $\lvert x+iy+2i\rvert=\lvert 2i(x+iy)-1\rvert$ $\lvert x+i(y+2)\rvert=\lvert(-1-2y)+2xi\rvert$ $\sqrt{x^2+(y+2)^2}=\sqrt{(-1-2y)^2+4x^2}$ $x^2+(y+2)^2=(-1-2y)^2+4x^2$ $3x^2+3y^2=3$ $x^2+y^2=1$ Unit circle i.e. circle centre $(0,0)$ radius 1
Inequalities	• $\lvert Re(z)\rvert\leq\lvert z\rvert$ and $\lvert Im(z)\rvert\leq\lvert z\rvert$ • $\lvert z+w\rvert\leq\lvert z\rvert+\lvert w\rvert$ • $\lvert z+w\rvert\geq\lvert z\rvert-\lvert w\rvert$ • $\lvert e^{real}\rvert=e^{real}$	

3 Forms

Cartesian Form
$z = x + iy$, with $x, y \in \mathbb{R}$

All above methods and properties have dealt with this form

Modulus Argument Form (aka Polar form)
$z = r(\cos\theta + i\sin\theta)$ where, $|z| = r$ and $\arg z = \theta$

Basic Properties:

- $z^* = \bar{z} = r(\cos\theta - i\sin\theta)$
- $z^n = r^n(\cos n\theta + i\ \sin n\theta)$ by De Moivre's Theorem
- $z^{-n} = r^{-n}(\cos n\theta - i\ \sin n\theta)$ by De Moivre's Theorem

Multiplying and Dividing Properties:

Mod Results

- $|z_1 z_2| = |z_1||z_2|$
- $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

Arg Results

- $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$.
  This works like indices rules. When we multiply, we add the powers.
  So $[r_1(\cos\theta_1 + i\sin\theta_1)][r_2(\cos\theta_2 + i\sin\theta_2)]$ can be multiplied quickly and is
  $r_1 r_2[\cos(\theta_1 + \theta_2)\ \ + i\sin(\theta_1 + \theta_2)] = r_1 r_2 e^{i(\theta_1+\theta_2)}$
- $arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$
  This works like indices rules. When we divide. we subtract the powers.
  So $\frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)}$ can be divided quickly and is $\frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$

Useful Results:

- $z + \frac{1}{z} = 2\cos\theta$
- $z - \frac{1}{z} = 2i\sin\theta$
- $z^n + \frac{1}{z^n} = z^n + z^{-n} = 2\cos n\theta$
  rearranging $\Rightarrow \cos n\theta = \frac{z^n + z^{-n}}{2}$
- $z^n - \frac{1}{z^n} = z^n - z^{-n} = 2i\sin n\theta$
  rearranging $\Rightarrow\ \sin n\theta = \frac{z^n - z^{-n}}{2i}$

Euler's Form
$re^{i\theta}$

Basic Properties:

- $z = re^{i\theta} = r(\cos\theta + i\sin\theta)$
- $z^* = \bar{z} = re^{-i\theta} = r(\cos\theta - i\sin\theta)$
- $z^n = r^n e^{in\theta} = r^n(\cos n\theta + i\ \sin n\theta)$
- $z^{-n} = r^{-n}e^{-in\theta} = r^{-n}(\cos n\theta - i\ \sin n\theta)$

Converting Between The 3 Forms

Modulus Argument→Cartesian
$\boldsymbol{r(cos\theta + isin\theta) \to a + bi}$:

Work out values of the trig function and this will immediately be in cartesian form

Example:

Convert $2\left(cos\frac{\pi}{3} + isin\frac{\pi}{3}\right)$ into cartesian form

$= 2\left(\left(\frac{1}{2}\right) + i\left(\frac{\sqrt{3}}{2}\right)\right)$

$= 1 + \sqrt{3}i$

Cartesian→Modulus Argument
$\boldsymbol{a + bi \to\ r(cos\theta + isin\theta)}$

Need to find r *and* θ. To find r and θ we use the fomrula

$$a + bi \Rightarrow \begin{cases} r = \sqrt{a^2 + b^2} \\ \theta = tan^{-1}\left(\left|\frac{b}{a}\right|\right) \end{cases}$$

and then draw the angle θ in the quadrant where the complex number $a + bi$ lies

Read off θ by starting on the positive x axis (like when you solve for trig using the CAST diagram)

(remember: $-\pi \le \theta < \pi$)

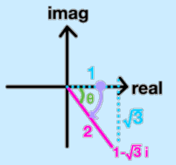
Or use the formula $\theta = \begin{cases} \tan^{-1}\left(\frac{b}{a}\right) & (if\ quadrant\ 1\ or\ 4) \\ \tan^{-1}\left(\frac{b}{a}\right) + \pi & (if\ quadrant\ 2) \\ \tan^{-1}\left(\frac{b}{a}\right) - \pi & (if\ quadrant\ 3) \end{cases}$

Sub this r and θ found into $r(cos\theta + isin\theta)$

Example:

Convert $z = 1 - \sqrt{3}i$ into modulus argument form

$r = \sqrt{(1)^2 + (-\sqrt{3})^2} = \sqrt{4} = 2$



$\theta = tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = \frac{\pi}{3}$

Complex number is in 4th quadrant so angle is $-\frac{\pi}{3}$

Plugging this into the form we get

$2\left(\cos\left(-\frac{\pi}{3}\right) + isin\left(-\frac{\pi}{3}\right)\right)$

$= 2\left(\cos\left(\frac{\pi}{3}\right) - isin\left(\frac{\pi}{3}\right)\right)$

Modulus Argument→Euler
$\boldsymbol{r(\cos\theta + i\sin\theta) \to re^{i\theta}}$

Get into Modulus Argument form if given cartesian form and then you can read Euler's form straight off by locating r and θ

This is easy since we already know r and θ from modulus argument form

Example:

Convert $2\left(cos\frac{\pi}{3} + isin\frac{\pi}{3}\right)$ into Euler form

$= 2e^{i\frac{\pi}{3}}$

Cartesian→Euler
$\boldsymbol{a + bi \to\ re^{i\theta}}$

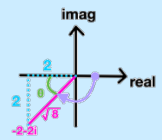
Turn this into modulus argument form using second column and then see third column

Example:

Convert $-2 - 2i$ into Euler Form

$r = \sqrt{(-2)^2 + (-2)^2} = \sqrt{8} = 2\sqrt{2}$

$\theta = tan^{-1}\left(\frac{2}{2}\right) = \frac{\pi}{4}$



$\theta = -\pi - \frac{3\pi}{4}$

Complex number is in 3rd quadrant so angle is $-\frac{3\pi}{4}$

Plugging this into the form we get

$2\sqrt{2}\left(\cos\left(-\frac{3\pi}{4}\right) + isin\left(-\frac{3\pi}{4}\right)\right)$

$= 2\sqrt{2}e^{-i\frac{3\pi}{4}}$

We commonly use mod/arg form to find the roots of complex numbers. Finding square roots is easy since we can compare coefficients instead like in example 3 under solving with complex numbers section, but when we get to third roots and above the algebra becomes messy and it is easier to put into modulus argument form and use De Moivre's

Example:

Find the cube roots of $8i$

$z^3 = 8i \Leftrightarrow\ z = (8i)^{\frac{1}{3}}$

Let's turn $8i$ into modulus argument form: r= $\sqrt{8^2} = 8$, $\theta = tan^{-1}\left(\frac{8}{0}\right) = \frac{\pi}{2}$

$z = \left(8\left(cos\left(\frac{\pi}{2} + 2n\pi\right) + isin\left(\frac{\pi}{2} + 2n\pi\right)\right)\right)^{\frac{1}{3}} = 8^{\frac{1}{3}}\left(cos\left(\frac{\pi}{6} + \frac{2n\pi}{3}\right) + isin\left(\frac{\pi}{6} + \frac{2n\pi}{3}\right)\right)$

$z = 2\left(cos\left(\frac{\pi}{6} + \frac{2n\pi}{3}\right) + isin\left(\frac{\pi}{6} + \frac{2n\pi}{3}\right)\right)$

Choose 3 values for n

Let $n = -1$:$2\left(cos\left(\frac{\pi}{6} - \frac{2\pi}{3}\right) + isin\left(\frac{\pi}{6} - \frac{2\pi}{3}\right)\right) = 2\left(cos\left(\frac{-3\pi}{6}\right) + isin\left(\frac{-3\pi}{6}\right)\right) = 2\left(cos\left(\frac{\pi}{2}\right) - isin\left(\frac{\pi}{2}\right)\right) = 2e^{-\frac{\pi}{2}}$

Let $n = 1$: $2\left(cos\left(\frac{\pi}{6} + \frac{2\pi}{3}\right) + isin\left(\frac{\pi}{6} + \frac{2\pi}{3}\right)\right) = 2\left(cos\left(\frac{5\pi}{6}\right) + isin\left(\frac{5\pi}{6}\right)\right) = 2e^{\frac{5\pi}{6}}$

Let $n = 0$: $2\left(cos\left(\frac{\pi}{6}\right) + isin\left(\frac{\pi}{6}\right)\right) = 2e^{\frac{\pi}{6}}$

These roots are equispaced around a circle of radius r (see the diagram on the right)